

On the quasisymmetric Hölder-equivalence problem for Carnot groups

P. Pansu*

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Abstract

A variant of Gromov's Hölder-equivalence problem, motivated by a pinching problem in Riemannian geometry, is discussed. A partial result is given. The main tool is a general coarea inequality satisfied by packing energies of maps.

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1 The problem

1.1 The Hölder equivalence problem

In [G], M. Gromov conjectured that if there exists a C^α homeomorphism from an open set in Euclidean space \mathbb{R}^3 to Heisenberg group equipped its Carnot-Carathéodory metric, then $\alpha \leq \frac{1}{2}$. This is still open. Gromov proved the upper bound $\alpha \leq \frac{2}{3}$, which has not been improved since, in spite of many efforts, [Z1], [LDZ], [Z2].

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More generally, let X be a metric space which is a topological manifold. Let $\alpha(X)$ be the supremum of α such that there exists a C^α homeomorphism from an open set in Euclidean space $\mathbb{R}^{\dim(X)}$ to X .

The case of Carnot groups equipped with Carnot-Carathéodory metrics is especially interesting, since they are prototypes of isometry-transitive and self-similar geodesic metric spaces. Let \mathfrak{g} be a Carnot Lie algebra, i.e. a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \mathfrak{g}_r$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and such that \mathfrak{g}_1 generates \mathfrak{g} . Let G denote the corresponding simply connected Lie group. A choice of Euclidean norm on \mathfrak{g}_1 determines a left-invariant Carnot-Carathéodory metric on G , of Hausdorff dimension

$$Q = \sum_{i=1}^r i \dim(\mathfrak{g}_i).$$

Any two such metrics are bi-Lipschitz equivalent, therefore the following exponent is well-defined: let $\alpha(G)$ be the supremum of α such that there exists a C^α homeomorphism from an open set in Euclidean space $\mathbb{R}^{\dim(G)}$ to G . The exponential map is $\frac{1}{r}$ -Hölder continuous, showing that $\alpha(G) \geq \frac{1}{r}$.

Gromov has developed numerous tools to get lower bounds on α , and has obtained the following results, among others.

Theorem (Gromov 1996). *Let X be a metric space which is a topological manifold of dimension n and Hausdorff dimension Q . Then $\alpha(X) \leq \frac{n}{Q}$.*

Let G be a Carnot group of topological dimension n and Hausdorff dimension Q . Then $\alpha(G) \leq \frac{n-1}{Q-1}$.

Let $G = \text{Heis}_{\mathbb{C}}^m$ be the m -th Heisenberg group, of dimension $2m+1$. Then $\alpha(\text{Heis}_{\mathbb{C}}^m) \leq \frac{m+1}{m+2}$.

1.2 The quasimetric Hölder equivalence problem

We address a variant of the Hölder equivalence problem motivated by Riemannian geometry.

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ -pinched if sectional curvature ranges between $-a$ and δa for some $a > 0$. Define the *optimal pinching* $\delta(M)$ of M as the least $\delta \geq -1$ such that M is quasiisometric to a δ -pinched complete simply connected Riemannian manifold.

There are a few homogeneous Riemannian manifolds whose optimal pinching is known. [P3] deals with semi-direct products $\mathbb{R} \ltimes \mathbb{R}^n$ where \mathbb{R} acts on \mathbb{R}^n by matrices with only two distinct eigenvalues.

Optimal pinching is not known for the oldest examples, rank one symmetric spaces of noncompact type. These are hyperbolic spaces over the reals $H_{\mathbb{R}}^n$, the complex numbers $H_{\mathbb{C}}^m$, the quaternions $H_{\mathbb{H}}^m$, and the octonions $H_{\mathbb{O}}^2$. Real hyperbolic spaces have sectional curvature -1 , and therefore optimal pinching -1 . All other rank one symmetric spaces are $-\frac{1}{4}$ -pinched. The optimal pinching of $H_{\mathbb{C}}^m$, $H_{\mathbb{H}}^m$ or $H_{\mathbb{O}}^2$ is conjectured to be $-\frac{1}{4}$, but still unknown. [P4] suggests a different, yet nonconclusive, approach in the case of $H_{\mathbb{C}}^2$.

Negatively curved manifolds M come with an ideal boundary ∂M , equipped with a family of (pairwise equivalent) *visual metrics*. If curvature is ≤ -1 , visual metrics are true distances, i.e. satisfy triangle inequality. Polar coordinates are defined globally, extending into a homeomorphism from the round sphere to the ideal boundary, $\exp : S \rightarrow \partial M$. Rauch's comparison theorem shows that the behaviour of visual metrics in such coordinates reflects the distribution of sectional curvature: if M is δ -pinched, \exp is distance increasing and C^α -Hölder continuous with $\alpha = \sqrt{-\delta}$.

For instance, ideal boundaries of rank one symmetric spaces are round spheres or subRiemannian manifolds quasisymmetrically equivalent to Heisenberg groups $Heis_{\mathbb{C}}^{m-1}$, $Heis_{\mathbb{H}}^{m-1}$, $Heis_{\mathbb{O}}^1$.

Finally, a quasiisometry between negatively curved manifolds M and M' extends to a quasisymmetric homeomorphism between ideal boundaries $\partial M \rightarrow \partial M'$.

Piling things up, a quasiisometry of a rank one symmetric space M with a δ -pinched manifold M' provides us with a metric space $X' = \partial M'$ which is quasisymmetrically equivalent to a sub-Riemannian manifold and a $C^{\sqrt{-\delta}}$ homeomorphism from the round sphere $\exp : S \rightarrow X'$. Furthermore, \exp^{-1} is Lipschitz.

Definition 1. *Let X be a metric space which is a topological manifold. Let $\alpha_{qs}(X)$ be the supremum of α such that there exists a metric space X' quasisymmetric to an open subset of X and a homeomorphism of an open subset of Euclidean space to X' which is C^α Hölder continuous, and whose inverse is Lipschitz.*

Question 1 (Quasisymmetric Hölder-Lipschitz equivalence problem). *Let G be a non abelian Carnot group. Prove that $\alpha_{qs}(G) \leq \frac{1}{2}$.*

The fact that Carnot groups of Hausdorff dimension Q have conformal dimension Q (see [P1]) implies that $\alpha_{qs}(G) \leq \frac{n}{Q}$. We make one step further.

Theorem 2. *Let G be a Carnot group of topological dimension n and Hausdorff dimension Q . Then $\alpha_{qs}(G) \leq \frac{n-1}{Q-1}$.*

This has an unsharp consequence for the pinching problem.

Corollary 3. *If $H_{\mathbb{C}}^2$ is quasiisometric to a δ -pinched Riemannian manifold, then $\delta \geq -\frac{4}{9}$.*

There are similar conclusions for other rank one symmetric spaces.

2 The method

Following Gromov, [G], we manage to produce a smooth hypersurface in \mathbb{R}^n whose image in the unknown space X' has Hausdorff dimension at least $Q - 1$. If X' is a Carnot group, one knows, thanks to the isoperimetric inequality, that every hypersurface has Hausdorff dimension at least $Q - 1$. The isoperimetric inequality or the Hausdorff dimension of subsets are not quasisymmetry invariants, so a different argument is needed.

Ours is based on a consequence of the coarea formula. If X is a connected open subset of a Carnot group of Hausdorff dimension Q , and $u : X \rightarrow \mathbb{R}$ is a Lipschitz function, then (see [M]),

$$\int_X |\nabla u|^Q \leq \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} |\nabla u|^{Q-1} \right) dt \leq \text{const.} \int_{\mathbb{R}} \mathcal{H}^{Q-1}(u^{-1}(t)) dt. \quad (1)$$

Here, ∇u denotes the horizontal gradient. Since, for non constant u , $\int_X |\nabla u|^Q > 0$, this shows that there exists $t \in \mathbb{R}$ such that $\mathcal{H}^{Q-1}(u^{-1}(t)) > 0$, and therefore $u^{-1}(t)$ has Hausdorff dimension at least $Q - 1$.

On the unknown metric space X' , we define, mimicking Tricot's construction of packing measures, quasisymmetry invariant analogues of the conformally invariant integrals $\int_X |\nabla u|^Q$ and $\int_{u^{-1}(t)} |\nabla u|^{Q-1}$. The sequence of inequalities (1) persists (packing measures instead of covering measures are required in order to have the inequality in the appropriate direction). The main point is non vanishing of the packing analogue of $\int_X |\nabla u|^Q$. This is proved by a coarse analogue of the length area method (the covering version of which dates back to the 1980's, [P1]).

There is some hope to improve exponents and reach the sharp exponent $\frac{1}{2}$, at least for the simpler Hölder-Lipschitz equivalence problem, see the last section.

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3 Packings

3.1 Bounded multiplicity packings

Let X be a metric space. A ball B in X is the data of a point $x \in X$ and a radius $r \geq 0$. For brevity, we also denote $B(x, r)$ by B . If $\lambda \geq 0$, λB denotes $B(x, \lambda r)$.

Definition 4. Let N be an integer, let $\ell \geq 1$. Let X be a metric space. An ℓ -packing is a countable collection of balls $\{B_j\}$ such that concentric balls ℓB_j are pairwise disjoint. An (N, ℓ) -packing is a collection of balls $\{B_j\}$ which is the union of at most N ℓ -packings.

3.2 Covering and packing measures

Definition 5. Let ϕ be a positive function on the set of balls in X . Two types of (pre-)measures can be obtained from it as follows. Let A be a subset of X , $\epsilon > 0$ and $p \geq 1$.

1. *Covering measure*

$$K\Phi_{N,\ell}^{p;\epsilon}(A) = \inf \left\{ \sum_i \phi(B_i)^p \quad ; \quad \{B_i\} (N, \ell) \text{--packing of } X \text{ that covers } A, \right. \\ \left. B_i \text{ of diameter } \leq \epsilon \right\},$$

$$K\Phi_{N,\ell}^p(A) = \lim_{\epsilon \rightarrow 0} K\Phi_{N,\ell}^{p;\epsilon}(A).$$

2. Packing measure

$$P\Phi_{N,\ell}^{p;\epsilon}(A) = \sup\left\{\sum_i \phi(B_i)^p \quad ; \quad \{B_i\}(N, \ell) - \text{packing of } X, \right. \\ \left. B_i \text{ centered on } A, \text{ of diameter } \leq \epsilon\right\},$$

$$P\Phi_{N,\ell}^p(A) = \lim_{\epsilon \rightarrow 0} P\Phi_{N,\ell}^{p;\epsilon}(A).$$

Remark 6. $K\Phi_{N,\ell}^p$ is a measure. $P\Phi_{N,\ell}^p$ is merely a pre-measure. To turn it into a measure, it suffices to force σ -additivity by declaring

$$\tilde{P}\Phi_{N,\ell}^p(A) = \inf\left\{\sum_{j \in \mathbb{N}} P\Phi_{N,\ell}^p(A_j); A \subset \bigcup_{j \in \mathbb{N}} A_j\right\}.$$

We shall ignore this point, we do not need $P\Phi_{N,\ell}^p$ to be a measure.

Example 7. Take

$$\phi(B) = \delta(B) = \text{radius}(B).$$

The resulting covering measure $K\Delta_{N,\ell}^p$ is a minor variant of Hausdorff spherical measure. The packing pre-measure $P\Delta_{N,\ell}^p$ is called here the p -dimensional packing pre-measure with parameters (N, ℓ) .

Definition 8. Let $N \in \mathbb{N}$, $\ell \geq 1$. Let X be a metric space. The packing dimension of a subset $A \subset X$ is

$$\dim_{N,\ell}(A) = \sup\{p; P\Delta_{N,\ell}^p(A) > 0\}.$$

Example 9. Let X be Euclidean space or a Carnot group. Let A be a horizontal line segment of length L . For all $N \geq 1$ and $\ell \geq 1$, $P\Delta_{N,\ell}^{p;\epsilon}(A) \sim L\epsilon^{p-1}$. It follows that $\dim_{N,\ell}(A) = 1$.

Indeed, for any (N, ℓ) -packing centered on A , $\sum \text{radius}(B_i) \leq \frac{1}{2}NL$, so, if balls have radii $\leq \epsilon$,

$$\sum \text{radius}(B_i)^p \leq \frac{1}{2}NL\epsilon^{p-1}.$$

Conversely, there is an $(N, 1)$ -packing by ϵ -balls which achieves this bound.

Example 10. Let X be Heisenberg group. Let A be a vertical line segment of height h . For all $N \geq 1$ and $\ell \geq 1$, $P\Delta_{N,\ell}^{p;\epsilon}(A) \sim h\epsilon^{p-2}$. It follows that $\dim_{N,\ell}(A) = 2$.

Indeed, for any (N, ℓ) -packing centered on A , $\sum \text{radius}(B_i)^2 \leq cNh$, where c is the constant such that the height of an r -ball equals $\sqrt{\frac{r}{c}}$, so, if balls have radii $\leq \epsilon$,

$$\sum \text{radius}(B_i)^p \leq cNh\epsilon^{p-2}.$$

Conversely, there is an $(N, 1)$ -packing by ϵ -balls which achieves this bound.

Example 11. Let X and X' be open subsets of Carnot groups G and G' of Hausdorff dimensions Q and Q' . Let $u : G \rightarrow G'$ be a surjective homogeneous homomorphism, with kernel V . Let \mathcal{L} denote the Lebesgue measure on V . Let A be a bounded open subset of V . Assume that the relative boundary ∂A of A in V has vanishing Lebesgue measure $\mathcal{L}(\partial A) = 0$. For all $\ell \geq 1$ and large enough N , $P\Delta_{N,\ell}^{p;\epsilon}(A) \sim \mathcal{L}(A)\epsilon^{p-Q+Q'}$. It follows that $\dim_{N,\ell}(A) = Q - Q'$.

Indeed, by translation and dilation invariance, if B is an r -ball centered on A , the Lebesgue measure $\mathcal{L}(A \cap B) = \frac{1}{c} r^{Q-Q'}$ for some constant $c > 0$. For any (N, ℓ) -packing by balls centered on A , of radius $\leq \epsilon$,

$$\sum \text{radius}(B_i)^{Q-Q'} \leq cN\mathcal{L}(A + \epsilon),$$

where $A + \epsilon$ denotes the ϵ -neighborhood of A in V . Thus

$$\sum \text{radius}(B_i)^p \leq cN\mathcal{L}(A + \epsilon)\epsilon^{p-Q+Q'}.$$

Conversely, for large enough N , there is an $(N, 1)$ -packing by ϵ -balls which achieves asymptotically this bound.

3.3 Quasisymmetric invariance

Proposition 12. Let X, X' be metric spaces. Let $f : X \rightarrow X'$ be a quasisymmetric homeomorphism. Given a function ψ on balls of X' , define a function ϕ on balls of X as follows. Given a ball B of X centered at x , let B' be the smallest ball centered at $f(x)$ containing $f(B)$, and let $\phi(B) = \psi(B')$. Then, for all $\ell' \geq 1$, there exists $\ell \geq 1$ such that for all $N \in \mathbb{N}$, all $p > 0$, and all subsets $A \subset X$,

$$\begin{aligned} K\Psi_{N,\ell'}^p(f(A)) &\leq K\Phi_{N,\ell}^p(A), \\ P\Phi_{N,\ell}^p(A) &\leq P\Psi_{N,\ell'}^p(f(A)). \end{aligned}$$

Symmetric statements also hold since f^{-1} is quasisymmetric as well.

Proof Let $\eta : (0, +\infty) \rightarrow (0, +\infty)$ be the homeomorphism measuring the quasisymmetry of f , i.e. for every triple x, y, z of distinct points of X ,

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

Fix $\ell' \geq 1$ and $N \in \mathbb{N}$. Let B be a ball of X . The procedure above defines a corresponding B' of radius ρ' . Let $y \in B$ be such that $d(f(x), f(y)) = \rho'$. If $z \in f^{-1}(\ell' B')$, $d(f(x), f(z)) \leq \ell' \rho'$, so

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \geq \frac{\rho'}{\ell' \rho'} = \frac{1}{\ell'}.$$

By quasi-symmetry, this implies that $\eta\left(\frac{d(x, y)}{d(x, z)}\right) \geq \frac{1}{\ell'}$, and thus

$$d(x, z) \leq \frac{1}{\eta^{-1}\left(\frac{1}{\ell'}\right)} d(x, y).$$

In other words, $z \in \ell B$ with $\ell = \frac{1}{\eta^{-1}(\frac{1}{\ell'})}$. We conclude that, for every $\lambda' \geq 1$, there exist $\lambda \geq 1$ such that $f(B) \subset B'$ and $f^{-1}(\lambda' B') \subset \lambda B$. We apply it with $\lambda' = 3\ell'$ to get $\ell = \lambda \geq 1$.

We show that (N, ℓ) -packings are mapped to (N, ℓ') -packings by contradiction. Let $\{B_j\}$ be an (N, ℓ) -packing of X . Let B'_1, \dots, B'_N be distinct balls from the corresponding packing of X' , ordered by increasing radii. Assume that $\bigcap_{j=1}^N \ell' B'_j$ is non empty. Since $\ell' B'_1$ and $\ell' B'_j$ intersect and $\ell' B'_1$ is smaller, $\ell' B'_1 \subset 3\ell' B'_j$, so $f(x_1)$ belongs to $\bigcap_{j=1}^N 3\ell' B'_j$. Then

$$x_1 \in f^{-1}\left(\bigcap_{j=1}^N 3\ell' B'_j\right) \subset \bigcap_{j=1}^N \ell B_j = \emptyset,$$

contradiction. We conclude that $\{B'_j\}$ is an (N, ℓ') -packing of X' , for $\ell = \frac{1}{\eta^{-1}(\frac{1}{3\ell'})}$. If $\{B_i\}$ covers a subset A , then $\{B'_i\}$ covers $f(A)$. If $\{B_i\}$ is centered on A , then $\{B'_i\}$ is centered on $f(A)$. So the announced inequalities follow from taking an infimum or a supremum. ■

3.4 Hölder covariance

Proposition 13. *Let X, Y be metric spaces. Let $0 < \alpha \leq 1$. Let $N \in \mathbb{N}$ and $\ell \geq 1$. Let $f : X \rightarrow Y$ be C^α -Hölder continuous. Then, for every subset $A \subset X$,*

$$P\Delta_{N,\ell}^{\alpha p}(A) \geq \text{const. } P\Delta_{N,\ell}^p(f(A)).$$

It follows that $\dim_{N,\ell}(A) \geq \alpha \dim_{N,\ell}(f(A))$.

Proof Let $\{B'_i\}$ be an (N, ℓ) -packing of Y by balls of radii $\leq \epsilon$ centered on $f(A)$. By assumption, for all $x, x' \in X$,

$$d(f(x), f(x')) \leq C d(x, x')^\alpha.$$

If $B'_i = B(x'_i, r'_i)$, pick an inverse image $x_i \in f^{-1}(x'_i) \cap A$ and set

$$B_i = B(x_i, \frac{1}{\ell} (\frac{\ell r'_i}{C})^{1/\alpha}).$$

Then B_i are balls centered on A , of radii $\leq \epsilon' := \frac{1}{\ell} (\frac{\ell \epsilon}{C})^{1/\alpha}$. Furthermore, the collection of sets

$$f(\ell B_i) \subset \ell B'_i$$

has multiplicity $< N$, so the collection of balls $\{B_i\}$ is an (N, ℓ) -packing of X . By definition,

$$\begin{aligned} P\Delta_{N,\ell}^{\alpha p; \epsilon'}(A) &\geq \sum_i \text{radius}(B_i)^{\alpha p} \\ &= \sum_i \left(\frac{1}{\ell} \left(\frac{\ell r'_i}{C}\right)^{1/\alpha}\right)^{\alpha p} \\ &= \ell^{p(1-\alpha)} C^{-p} \sum_i r'^p_i. \end{aligned}$$

Taking the supremum over (N, ℓ) -packings yields

$$P\Delta_{N,\ell}^{\alpha p; \epsilon'}(A) \geq \ell^{p(1-\alpha)} C^{-p} P\Delta_{N,\ell}^{p; \epsilon}(f(A)).$$

Letting ϵ tend to 0, one concludes

$$P\Delta_{N,\ell}^{\alpha p}(A) \geq \ell^{p(1-\alpha)} C^{-p} P\Delta_{N,\ell}^p(f(A)).$$

■

4 The coarea inequality

4.1 Energy

Definition 14. Let X be a metric space. Let $u : X \rightarrow M$ be a map to an auxiliary measure space (M, μ) . Let

$$e_u(B) = \mu(u(B)).$$

The total mass $PE_{u,N,\ell}^p(X)$ of the resulting packing pre-measure is the p -energy of u with parameters (N, ℓ) .

This is again a quasisymmetry invariant. If $f : X \rightarrow X'$ is a quasisymmetric homeomorphism, a map $u : X \rightarrow M$ gives rise to a map $u' = u \circ f^{-1} : X' \rightarrow M$. As in subsection 3.3, given a ball B of X centered at x , let B' be the smallest ball centered at $f(x)$ containing $f(B)$. Then $B \subset f^{-1}(B')$, $e_{u'}(B') = \mu(u(f^{-1}(B'))) \geq \mu(u(B)) = e_u(B)$, so we can assert that, for all $\ell' \geq 1$, there exists $\ell \geq 1$ such that for all $N \in \mathbb{N}$, all $p > 0$ and all subsets $A \subset X$,

$$PE_{u,N,\ell}^p(A) \leq PE_{u',N,\ell'}^p(f(A)).$$

Conversely, for all $\ell \geq 1$, there exists $\ell' \geq 1$ such that for all $N \in \mathbb{N}$, all $p > 0$ and all subsets $A \subset X$,

$$PE_{u',N,\ell'}^p(f(A)) \leq PE_{u,N,\ell}^p(A).$$

4.2 Coarea inequality

Proposition 15. Let X be a metric space. Let $u : X \rightarrow M$ be a map to a measure space (M, μ) . Let $N \in \mathbb{N}$ and $\ell \geq 1$. Then

$$PE_{u,N,2\ell}^p(X) \leq \int_M PE_{u,N,\ell}^{p-1}(u^{-1}(m)) d\mu(m).$$

Proof Let $\{B_i\}$ be an $(N, 2\ell)$ -packing of X consisting of balls with radius $\leq \epsilon$. Write

$$\mu(u(B_i)) = \int_M 1_{u(B_i)}(m) dm.$$

Furthermore, when $m \in u(B_i)$, pick a point $x_i \in B_i \cap u^{-1}(m)$ and let $B_{i,m}$ be the smallest ball centered at x_i which contains B_i . Note that

$B_{i,m} \subset 2B_i$, so that, for each $m \in M$, the collection $\{B_{i,m}\}$ is an (N, ℓ) -packing of X consisting of balls with radius $\leq 2\epsilon$ centered on $u^{-1}(m)$.

$$\begin{aligned}
\sum_i \mu(u(B_i))^p &= \sum_i \left(\int_M 1_{u(B_i)}(m) d\mu(m) \right) \mu(u(B_i))^{p-1} \\
&= \int_M \left(\sum_i 1_{u(B_i)}(m) \mu(u(B_i))^{p-1} \right) d\mu(m) \\
&= \int_M \left(\sum_{\{i; m \in u(B_i)\}} \mu(u(B_i))^{p-1} \right) d\mu(m) \\
&\leq \int_M \left(\sum_{\{i; m \in u(B_i)\}} \mu(u(B_{i,m}))^{p-1} \right) d\mu(m) \\
&\leq \int_M PE_{u,N,\ell}^{p-1;2\epsilon}(u^{-1}(m)) d\mu(m).
\end{aligned}$$

Taking the supremum over $(N, 2\ell)$ -packings yields

$$PE_{u,N,2\ell}^{p;\epsilon}(X) \leq \int_M PE_{u,N,\ell}^{p-1;2\epsilon}(u^{-1}(m)) d\mu(m).$$

Letting ϵ tend to 0, one concludes that

$$PE_{u,N,2\ell}^p(X) \leq \int_M PE_{u,N,\ell}^{p-1}(u^{-1}(m)) d\mu(m).$$

■

Remark 16. *Covering measures satisfy the opposite inequality*

$$KE_{u,N,\ell}^p(X) \geq \int_M KE_{u,N,\ell}^{p-1}(u^{-1}(m)) d\mu(m).$$

5 Lower bounds on energy

5.1 Modulus estimate

This is a packing version of a classical coarse modulus estimate for covering measures.

Proposition 17. *Let X be a metric space. Let Γ be a family of subsets of X , equipped with a measure $d\gamma$. For each $\gamma \in \Gamma$, a probability measure m_γ is given on γ . Let $p \geq 1$, $N \in \mathbb{N}$, $\ell \geq 1$. Assume that there exist arbitrarily fine (N, ℓ) -packings of X that cover X . Assume that there exists a constant τ such that for every small enough ball B of X ,*

$$\int_{\{\gamma \in \Gamma; \gamma \cap B \neq \emptyset\}} m_\gamma(\gamma \cap \ell B)^{1-p} d\gamma \leq \tau.$$

Then, for every function ϕ on the set of balls of X ,

$$P\Phi_{N,\ell}^p(X) \geq \frac{1}{N^{p-1}\tau} \int_\Gamma K\Phi_{N,\ell}^1(\gamma)^p d\gamma.$$

Proof Let $\{B_i\}$ be an (N, ℓ) -packing of X by balls of radius $\leq \epsilon$ that covers X . Let 1_i be the function defined on Γ by

$$1_i(\gamma) = \begin{cases} 1 & \text{if } \gamma \cap B_i \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For each set γ , the balls such that $1_i(\gamma) = 1$ cover γ , thus

$$\begin{aligned} K\Phi_{N,\ell}^{1;\epsilon}(\gamma) &\leq \sum_i 1_i(\gamma)\phi(B_i) \\ &= \sum_i 1_i(\gamma)\phi(B_i)m_\gamma(\gamma \cap \ell B_i)^{\frac{1-p}{p}}m_\gamma(\gamma \cap \ell B_i)^{\frac{p-1}{p}}. \end{aligned}$$

Hölder's inequality gives

$$K\Phi_{N,\ell}^{1;\epsilon}(\gamma)^p \leq \left(\sum_i 1_i(\gamma)\phi(B_i)^p m_\gamma(\gamma \cap \ell B_i)^{1-p} \right) \left(\sum_i m_\gamma(\gamma \cap \ell B_i) \right)^{p-1}.$$

Since the covering $\{\ell B_i\}$ has multiplicity $< N$ and m_γ is a probability measure, the rightmost factor is $< N^{p-1}$. Integrating over Γ gives

$$\begin{aligned} \int_\Gamma K\Phi_{N,\ell}^{1;\epsilon}(\gamma)^p d\gamma &\leq N^{p-1} \sum_i \phi(B_i)^p \left(\int_\Gamma 1_i(\gamma)m_\gamma(\gamma \cap \ell B_i)^{1-p} d\gamma \right) \\ &\leq N^{p-1} \tau \sum_i \phi(B_i)^p \\ &\leq N^{p-1} \tau P\Phi_{N,\ell}^{p;\epsilon}(X). \end{aligned}$$

■

Remark 18. If X is doubling at small scales, fine covering (N, ℓ) -packings exist with N depending only on ℓ .

Indeed, pick a maximal packing by disjoint $\frac{\epsilon}{2}$ balls. Then the doubled balls cover. If two ℓ times larger balls $B(x, \ell\epsilon)$ and $B(x', \ell\epsilon)$ overlap, then $B(x', \frac{\epsilon}{2}) \subset B(x, (2\ell + 1)\epsilon)$. The number of such balls is bounded above by the number of disjoint $\frac{\epsilon}{2}$ -balls in a $(2\ell + 1)\epsilon$ -ball, which, in a doubling metric space, is bounded above in terms of ℓ only. Of course, we need this doubling property only for ϵ small.

Example 19. Let X be a Carnot group of Hausdorff dimension Q . Let Γ be a family of parallel horizontal unit line segments. Then, for all $\ell > 1$ and suitable N , the assumptions of Proposition 17 are satisfied with $p = Q$, m_γ the length measure, $d\gamma$ the Lebesgue measure on a codimension 1 subgroup.

Indeed, apply translation and dilation invariance.

Definition 20. Let Γ be a family of subsets of X . Say a function ϕ on the set of balls of X is Γ -admissible if for every $\gamma \in \Gamma$,

$$K\Phi_{N,\ell}^1(\gamma) \geq 1.$$

The (p, N, ℓ) -modulus of the family Γ is the infimum of $P\Phi_{N,\ell}^p(X)$ over all Γ -admissible functions ϕ .

Thus Proposition 17 states a sufficient condition for a family of subsets to have positive p -modulus. p -modulus is quasisymmetry invariant in the following sense. Let $f : X \rightarrow X'$ be a quasisymmetric homeomorphism. Then for all $\ell' \geq 1$, there exists $\ell \geq 1$ such that

$$M_{N,\ell}^p(\Gamma) \leq M_{N,\ell'}^p(f(\Gamma)),$$

and for all $\ell \geq 1$, there exists $\ell' \geq 1$ such that

$$M_{N,\ell'}^p(f(\Gamma)) \leq M_{N,\ell}^p(\Gamma).$$

Example 21. Let X be a Carnot group of Hausdorff dimension Q . For every $\ell \geq 1$, there exists $N \in \mathbb{N}$ such that families of parallel horizontal unit line segments have positive (Q, N, ℓ) -modulus.

5.2 Quasiconformal submersions

Definition 22. Let X and Y be metric spaces. Say a continuous map $u : X \rightarrow Y$ is a quasiconformal submersion if there exists a homeomorphism $\eta : (0, +\infty) \rightarrow (0, +\infty)$ such that for every ball B of X , there exists a ball B' of Y such that for all $\lambda \geq 1$,

$$B' \subset u(B) \subset u(\lambda B) \subset \eta(\lambda)B'.$$

This notion is quasisymmetry invariant, both on the domain and on the range.

Example 23. If balls in X are connected and $Y = \mathbb{R}$, any continuous map $X \rightarrow Y$ is a quasiconformal submersion.

By quasisymmetry invariance, this fact generalizes to spaces quasisymmetrically equivalent to metric spaces whose balls are connected.

Example 24. Let $X = G$ and $Y = G'$ be Carnot groups. Then any surjective homogeneous homomorphism $u : G \rightarrow G'$ is a quasiconformal submersion. More generally, any contact map between open sets of Carnot groups whose differential is continuous and surjective is locally a quasiconformal submersion.

Corollary 25. Let X be a connected open subset of a Carnot group of Hausdorff dimension Q . Let M be a metric measure space. Assume that there exist constants d and ν such that for all small enough balls in M ,

$$\mu(B(m, r)) \geq \nu r^d.$$

Let $u : X \rightarrow M$ be a nonconstant quasiconformal submersion. Then

$$PE_{u,N,\ell}^{Q/d}(X) > 0.$$

Proof Let $\phi = e_u^{1/d}$. Let η be the function measuring quasiconformality of u . For every ball B in X , $u(B)$ contains a ball of radius $\geq \frac{\text{diameter}(u(B))}{\eta(1)}$, thus

$$\phi(B) = \mu(u(B))^{1/d} \geq \frac{\nu^{1/d}}{\eta(1)} \text{diameter}(u(B)).$$

Let $\Gamma \subset X$ be a family of parallel horizontal line segments of equal lengths. If a collection $\{B_i\}$ of balls covers one such segment γ ,

$$\text{diameter}(u(\gamma)) \leq \sum_i \text{diameter}(u(\gamma \cap B_i)) \leq \frac{\eta(1)}{\nu^{1/d}} \sum_i \phi(B_i).$$

This shows that

$$\int_{\Gamma} K \Phi_{N,\ell}^1(\gamma)^p d\gamma \geq \frac{\nu^{1/d}}{\eta(1)} \int_{\Gamma} \text{diameter}(u(\gamma))^p d\gamma.$$

Assume by contradiction that $PE_{u,N,\ell}^{Q/d}(X) = P\Phi_{N,\ell}^Q(X) = 0$. Proposition 17 implies that u is constant on every segment $\gamma \in \Gamma$. This proves that u is constant on every horizontal segment, hence on every polygonal curve made of horizontal segments. Since such curves allow to travel from any point to any other point of X , u is constant, contradiction. \blacksquare

6 On the quasi-symmetric Hölder-Lipschitz problem

6.1 Energy dimension

Definition 26. Let X be a metric space. Define its energy dimension as the infimum of exponents p such that there exist non constant continuous functions $u : X \rightarrow \mathbb{R}$ with finite p -energy $PE_{u,N,\ell}^p(X)$ for $\ell = 1$ and all large enough N .

This is a quasisymmetry invariant. By definition, this is less than packing dimension. Think of it as an avatar of *conformal dimension* (a generic term for the infimum of all dimensions of metric spaces quasisymmetrically equivalent to X , see [P1], [MKT]).

Example 27. The energy dimension of an open subset of a Carnot group is equal to its Hausdorff dimension.

Indeed, since real valued functions are quasiconformal submersions, this follows from Corollary 25.

Lemma 28. Let X be a metric space of energy dimension Q . Let M be a d -Ahlfors regular metric space (at small scales). If $u : X \rightarrow M$ is a quasiconformal submersion, then for all $p < Q$, $PE_{u,N,\ell}^{p/d}(X) > 0$.

Proof d -Ahlfors regular at small scales means that for all small enough balls in M ,

$$\nu r^d \leq \mu(B(m, r)) \leq \frac{1}{\nu} r^d.$$

Under this assumption, there exists a constant c such that for all small enough balls B of X ,

$$\frac{1}{c} \text{diameter}(u(B)) \leq e_u(B)^{1/d} \leq c \text{diameter}(u(B)).$$

Let $p < Q$. We prove by contradiction that $PE_{u,N,\ell}^{p/d}(X) > 0$. Pick a point $m_0 \in M$. Define a real valued function v on X by

$$v(x) = d(m_0, u(x)).$$

For all small enough balls B in X ,

$$\text{diameter}(v(B)) \leq \text{diameter}(u(B)) \leq c e_u(B)^{1/d},$$

thus

$$PE_{v,N,\ell}^p(X) \leq c^p PE_{u,N,\ell}^{p/d} = 0.$$

By definition of energy dimension, this implies that v is constant. Since this holds for every $m_0 \in M$, one finds that u is constant, contradiction. One concludes that $PE_{u,N,\ell}^{p/d}(X) > 0$ for all $p < Q$. \blacksquare

6.2 Proof of the main theorem

Lemma 29. *Let $X \subset G$ and $M \subset G''$ be open subsets of Carnot groups of Hausdorff dimensions Q and Q'' . Let X' be a metric space. Let $f : X \rightarrow X'$ be a C^α -Hölder continuous homeomorphism. Assume that $f^{-1} : X' \rightarrow X$ is Lipschitz. Let $u : G \rightarrow G''$ be a surjective homomorphism mapping X to M . Let $u' = u \circ f^{-1} : X' \rightarrow M$. Assume that for all $p < \frac{Q'}{Q''}$, all $\ell > 1$ and N large enough, $PE_{u',N,2\ell}^p(X') > 0$. Then*

$$\alpha \leq \frac{Q - Q''}{Q' - Q''}.$$

Proof Fix $\ell > 1$ and choose N according to Example 18. For all $m \in M$, $\dim_{N,\ell}(u^{-1}(m)) = Q - Q''$ (Example 11). By assumption, for all $p < Q'/Q''$, $PE_{u',N,2\ell}^p(X') > 0$. Proposition 15 implies that there exists $m_p \in M$ such that $PE_{u',N,\ell}^{p-1}(u'^{-1}(m_p)) > 0$.

If f^{-1} is Lipschitz, so is u' , thus $e_{u'}(B) \leq \text{const. radius}(B)^{Q''}$, so $PE_{u',N,\ell}^{p-1} \leq \text{const. } P\Delta_{N,\ell}^{Q''(p-1)}$. It follows that $P\Delta_{N,\ell}^{Q''(p-1)}(u'^{-1}(m_p)) > 0$, and

$$\dim_{N,\ell}(u'^{-1}(m_p)) \geq Q''(p-1).$$

Since $u'^{-1}(m_p) = f(u^{-1}(m_p))$, Hölder covariance (Proposition 13) implies that $Q - Q'' \geq \alpha Q''(p-1)$. Since this holds for all $p < Q'/Q''$, $Q - Q'' \geq \alpha(Q' - Q'')$. \blacksquare

Theorem 30. *Let $X \subset G$ be a connected open subset of a Carnot group of Hausdorff dimension Q . Let X' be a metric space of energy dimension Q' . Assume that X' is quasisymmetric to a metric space whose balls are connected. Let $f : X \rightarrow X'$ be a C^α -Hölder continuous homeomorphism. Assume that $f^{-1} : X' \rightarrow X$ is Lipschitz. Then*

$$\alpha \leq \frac{Q-1}{Q'-1}.$$

Proof Let $u : X \rightarrow \mathbb{R}$ be the restriction to X of a non constant group homomorphism $G \rightarrow \mathbb{R}$. By assumption, $u' = u \circ f^{-1} : X' \rightarrow \mathbb{R}$ is automatically a quasiconformal submersion (Example 23). According to Lemma 28, for all $p < Q'$, $PE_{u', N, 2\ell}^p(X') > 0$. Lemma 29 implies that $Q - 1 \geq \alpha(Q' - 1)$. \blacksquare

Theorem 2 is the special case where $G = \mathbb{R}^n$ and X' is quasisymmetrically equivalent to an open subset of a Carnot group of Hausdorff dimension Q . By quasisymmetry invariance and Example 27, X' has energy dimension $\geq Q$, so all assumptions of Theorem 30 are satisfied.

7 Speculation

7.1 Energies and homeomorphisms

Lemma 29 applies successfully with $G'' = \mathbb{R}$, since we have some information on energy dimensions of Carnot groups. To get closer to the conjectured estimate $\alpha_{qs}(Heis_{\mathbb{C}}^1) \leq \frac{1}{2}$, one needs understand the energies of maps of Carnot groups G' to $G'' = \mathbb{R}^2$. 2-energy is especially relevant, since nonvanishing of 2-energy would lead to the sharp bound $\frac{1}{2}$.

Let us define the “ \mathbb{R}^2 -dimension” of a metric space as the infimum of exponents p such that there exist non constant continuous *open* maps $u : X \rightarrow \mathbb{R}^2$ with finite p -energy. The “ \mathbb{R}^2 -dimension” of Heisenberg group $Heis_{\mathbb{C}}^1$ is $\leq \frac{4}{3}$, thus much smaller than 2. Indeed, the projection $Heis_{\mathbb{C}}^1 \rightarrow \mathbb{R}^2$, $(x, y, z) \mapsto (y, z)$ whose fibers are horizontal line segments has finite $\frac{4}{3}$ -energy.

Hence one must better exploit the special characters of maps involved.

Question 2. Let $g : Heis_{\mathbb{C}}^1 \rightarrow \mathbb{R}^3$ be a (local) homeomorphism. Does there exist a nonzero linear map $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $u \circ g : Heis_{\mathbb{C}}^1 \rightarrow \mathbb{R}^2$ has positive 2-energy ?

Proposition 31. Let X' be an open subset of Heisenberg group $Heis_{\mathbb{C}}^1$. Let $g : X' \rightarrow \mathbb{R}^3$ be almost everywhere differentiable. Assume that for all linear maps $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $PE_{u \circ g, N, \ell}^2(X') = 0$. Then the differential $Dg \in Hom(Heis_{\mathbb{C}}^1, \mathbb{R}^3)$ has rank ≤ 1 almost everywhere.

This suggests the following strategy:

1. Let X be a metric space. Prove that the composition g of a quasisymmetric homeomorphism $Heis_{\mathbb{C}}^1 \rightarrow X$ and a Lipschitz map $X \rightarrow \mathbb{R}^3$ is almost everywhere differentiable and absolutely continuous on lines.
2. Prove that if Dg has rank ≤ 1 almost everywhere, then g cannot be a homeomorphism.

7.2 Proof of Proposition 31

Lemma 32. Let $X \subset G$, $X' \subset G'$ be open subsets of Carnot groups of Hausdorff dimensions Q and Q' . Let β denotes the unit ball in G' . If $h : G' \rightarrow G$ is a homogeneous homomorphism, define

$$J(h) = \frac{vol(h(\beta))}{vol(\beta)^{Q/Q'}},$$

if h is surjective, $J(h) = 0$ otherwise. Let $g : X' \rightarrow X$ be a map which is almost everywhere differentiable. Then, for all N and $\ell \geq 1$,

$$PE_{g,N,2\ell}^{Q'/Q}(X') \geq C(N, \ell) \int_{X'} J(D_y g)^{Q'/Q} dy.$$

Proof Let us first prove that at points $y \in X'$ where g is differentiable,

$$\lim_{r \rightarrow 0} \frac{\text{vol}(g(B(y, r)))}{\text{vol}(B(y, r))^{Q/Q'}} = J(D_y g).$$

Up to translating, one can assume that y is the identity element. Then $B(y, r) = \delta_r(\beta)$,

$$\frac{\text{vol}(g(B(y, r)))}{\text{vol}(B(y, r))^{Q/Q'}} = \frac{\text{vol}(\delta_{1/r} g \delta_r(\beta))}{\text{vol}(\beta)^{Q/Q'}}.$$

By definition of differentiability ([P2]), $\delta_{1/r} g \delta_r$ converge uniformly to $D_y g$, hence the indicatrix $1_{\delta_{1/r} g \delta_r(\beta)}$ converges pointwise to $1_{D_y g(\beta)}$ away from the null-set $D_y g(\partial\beta)$, dominated convergence applies, and volumes converge.

Let β_j denote the $1/j$ -ball in G' . Let $Y \subset X'$ be an open subset compactly contained in X' . For j large enough, define a measurable function η_j on Y as follows.

$$\eta_j(y) = j \sup_{z \in \beta_j} d(D_y g(z), g(y)^{-1} g(yz)).$$

This function measures the speed at which g is approximated by its differential. By assumption, as j tends to ∞ , η_j tends to 0 almost everywhere. According to Lusin's theorem, the convergence is uniform on a compact set Z whose complement has arbitrarily small measure.

It follows that, at points $y \in Z$,

$$\frac{\text{vol}(g(y\beta_j))}{\text{vol}(y\beta_j)^{Q/Q'}} \rightarrow J(D_y g)$$

uniformly. Therefore

$$\text{vol}(g(y\beta_j))^{Q'/Q} \sim J(D_y g)^{Q'/Q} \text{vol}(y\beta_j) \quad \text{as } j \rightarrow \infty$$

uniformly as y varies in Z . Up to replacing balls containing points of Z with twice larger balls centered at points of Z , one gets the same conclusion, with a loss of a power of 2, for balls containing a point of Z . Pick an (N, ℓ) -packing of X' of mesh $\leq \epsilon$, covering a subset of almost full measure of X' , and therefore a large part Z' of Z . Discard balls which do not intersect Z . For the remaining balls,

$$J(D_y g)^{Q'/Q} \text{vol}(B) \lesssim \text{vol}(g(B))^{Q'/Q},$$

therefore

$$\int_{Z'} J_\epsilon(y)^{Q'/Q} dy \lesssim N \sum_B \text{vol}(g(B))^{Q'/Q} \leq N PE_{N,\ell}^{Q'/Q}(X'),$$

where

$$J_\epsilon(z) = \inf_{B(z,\epsilon)} J(Dg).$$

As ϵ decreases to 0, J_ϵ increases and converges almost everywhere to $J(Dg)$, so, by monotone convergence,

$$\int_{Z'} J(D_y g)^{Q'/Q} dy \leq C(N, \ell) PE_{N, \ell}^{Q'/Q}(X').$$

Finally, the measure of $X' \setminus Z'$ is arbitrarily small, thus Z' can be replaced with X' . \blacksquare

Let g be a map from an open subset of Heisenberg group to \mathbb{R}^3 which is differentiable almost everywhere. Assume that Dg has rank 2 on a set Y of positive measure. At each point of Y , one of the three projections $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ to coordinate planes is surjective. Thus, for one of them, u , this holds for a subset of positive measure. At such points, $J(D(u \circ g)) > 0$, thus $PE_{u \circ g, N, \ell}^2(Y) > 0$.

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Pierre Pansu
Laboratoire de Mathématiques d'Orsay,
Univ. Paris-Sud, CNRS, Université Paris-Saclay
91405 Orsay, France.
e-mail: pierre.pansu@math.u-psud.fr